



Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa


On some questions of Drewnowski and Łuczak concerning submeasures on \mathbb{N}^\star

Rafał Filipów, Piotr Szuca*

Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

ARTICLE INFO

Article history:

Received 13 October 2009

Available online 9 June 2010

Submitted by B. Bongiorno

Keywords:

Submeasure on \mathbb{N}

Lower semicontinuous submeasure

Equivalence of submeasures

Ideal

Coideal

Zero-ideal

Analytic ideal

Maximal almost disjoint family

Maximal ideal

p-Point ultrafilter

ABSTRACT

For a given submeasure ϕ on \mathbb{N} a sequence $(A_n)_{n \in \mathbb{N}}$ of subsets of \mathbb{N} is called a ϕ -sequence if $\phi(\bigcup_{n \in \mathbb{N}} F_n) = 0$ for every choice of finite sets $F_n \subset A_n$ ($n \in \mathbb{N}$). We show an example of a submeasure ϕ which is not the lim sup of lower semicontinuous submeasures, but $\lim \phi(A_n) = 0$ for any ϕ -sequence $(A_n)_n$. Moreover, we show that it is enough to consider only decreasing sequences $(A_n)_n$ in the above. We also construct a submeasure on \mathbb{N} which is not the core of a σ -submeasure, but has the property that for every sequence $(A_n)_n$ of subsets of \mathbb{N} if $\lim \phi(A_n) = 0$ then there is a subsequence $(n_k)_k$ and finite sets $E_{n_k} \subset A_{n_k}$ such that $(A_{n_k} \setminus E_{n_k})_k$ is a ϕ -sequence. These answer questions of Drewnowski and Łuczak (2008) from [2].

© 2010 Elsevier Inc. All rights reserved.

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of all natural numbers. A function $\phi : \mathcal{P}(\mathbb{N}) \rightarrow [0, +\infty]$ is called a *submeasure* if $\phi(\emptyset) = 0$, ϕ is monotone (i.e. $A \subset B \Rightarrow \phi(A) \leq \phi(B)$) and ϕ is subadditive (i.e. $\phi(A \cup B) \leq \phi(A) + \phi(B)$). We always assume that $\phi(\mathbb{N}) > 0$ for a submeasure ϕ .

A submeasure ϕ is a σ -submeasure if it is countably subadditive.

We say that a submeasure ϕ is *dominated* by a submeasure ψ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\phi(A) < \varepsilon$ whenever $\psi(A) < \delta$. We say that submeasures ϕ and ψ are *equivalent* if ϕ is dominated by ψ and ψ is dominated by ϕ .

A submeasure ϕ is *lower semicontinuous* (or *lsc*, for short) if $\phi(A) = \lim_{n \rightarrow \infty} \phi(A \cap \{0, 1, \dots, n\})$ for every $A \subset \mathbb{N}$. We say that ϕ is the *lim sup of lsc submeasures* if there is a sequence of lsc submeasures $(\phi_n)_{n \in \mathbb{N}}$, such that $\phi(A) = \limsup_{n \rightarrow \infty} \phi_n(A)$ for every $A \subset \mathbb{N}$.

For a submeasure ϕ , by the *core* of ϕ we mean the submeasure ϕ^\bullet defined by $\phi^\bullet(A) = \lim_{n \rightarrow \infty} \phi(A \setminus \{0, 1, \dots, n\})$.

An *ideal* on \mathbb{N} is a family $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ which is closed under taking subsets and finite unions.

For a submeasure ϕ , we define the ideal $\mathcal{Z}(\phi) = \{A \subset \mathbb{N} : \phi(A) = 0\}$ of ϕ -zero sets. It is not difficult to see that if submeasures ϕ and ψ are equivalent then $\mathcal{Z}(\phi) = \mathcal{Z}(\psi)$.

[☆] The work of both authors was supported by grants BW 5100-5-0148-8 and BW 5100-5-0157-9.

* Corresponding author.

E-mail addresses: rfilipow@mat.ug.edu.pl (R. Filipów), pszuca@radix.com.pl (P. Szuca).

URL: <http://rfilipow.mat.ug.edu.pl/> (R. Filipów).

For a given submeasure ϕ on \mathbb{N} , a sequence $(A_n)_{n \in \mathbb{N}}$ of subsets of \mathbb{N} is called a ϕ -sequence if $\phi(\bigcup_{n \in \mathbb{N}} F_n) = 0$ for every choice of finite sets $F_n \subset A_n$ ($n \in \mathbb{N}$).

We say that a submeasure ϕ satisfies *condition*

- (A) if $\lim_{n \rightarrow \infty} \phi(A_n) = 0$ for every ϕ -sequence $(A_n)_{n \in \mathbb{N}}$;
- (B) if $\lim_{n \rightarrow \infty} \phi(A_n) = 0$ for every decreasing sequence $(A_n)_{n \in \mathbb{N}}$ such that there is no $Z \subset \mathbb{N}$ with $\phi(Z) > 0$ and $Z \setminus A_n$ finite for every $n \in \mathbb{N}$;
- (C) if for every sequence $(A_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} \phi(A_n) = 0$ there is a subsequence $(n_k)_{k \in \mathbb{N}}$ and finite sets $E_{n_k} \subset A_{n_k}$ such that $(A_{n_k} \setminus E_{n_k})_{k \in \mathbb{N}}$ is a ϕ -sequence.

In [2], the authors used conditions (A), (B) and (C) to show the equivalence of lsc submeasures ϕ and ψ for which $\mathcal{Z}(\phi^*) = \mathcal{Z}(\psi^*)$. They also showed that (A) implies (B), that the lim sup of a sequence of lsc submeasures has (A), and if a submeasure ϕ is the core of a σ -submeasure then ϕ has (C). They formulated main theorems of their paper using properties (A), (B) and (C). They asked the following questions.

- (1) Is condition (A) stronger than (B)?
- (2) Does there exist a submeasure with property (A) which is not equivalent to the limsup of a sequence of lsc submeasures?
- (3) Does there exist a submeasure with property (C) which is not equivalent to the core of a σ -submeasure?

We answer these questions in Sections 2, 3 and 4, respectively. The answer to question (3) is only partial (it needs some additional set theoretic assumption).

The authors of [2] focus their considerations on nonatomic submeasures (a submeasure ϕ is said to be *nonatomic* if for every $\varepsilon > 0$ there exists a finite partition A_0, A_1, \dots, A_{n-1} of \mathbb{N} with $\phi(A_i) \leq \varepsilon$ for each i). We answer questions (2) and (3) affirmatively, however our examples are not nonatomic. We do not know the answer to those questions if we additionally require that a submeasure is nonatomic.

2. The equivalence of properties (A) and (B)

We say that a submeasure ϕ satisfies *property (A')* if $\lim_{n \rightarrow \infty} \phi(A_n) = 0$ for every decreasing ϕ -sequence $(A_n)_{n \in \mathbb{N}}$.

Proposition 1. *Let ϕ be a submeasure. Then the following conditions are equivalent.*

- (1) ϕ satisfies (A).
- (2) ϕ satisfies (A').

Proof. The implication “(1) \Rightarrow (2)” is obvious. The implication (2) \Rightarrow (1) is an immediate consequence of

Fact 2. *For any ϕ -sequence $(A_n)_n$, also the sequence $(B_n)_n$, where $B_n = \bigcup_{i \geq n} A_i$, is a ϕ -sequence.*

To see this fact, let $F_k \subset B_k$, $k \in \mathbb{N}$, be finite sets. For any $k \leq n$ let $F'_{k,n} = F_k \cap A_n$. Note that $F_k = \bigcup_{n \geq k} F'_{k,n}$ and, if we denote $E_n = \bigcup_{k \leq n} F'_{k,n}$, then E_n is finite and $E_n \subset A_n$. Since $(A_n)_n$ is a ϕ -sequence and $\bigcup_{n \in \mathbb{N}} F_n = \bigcup_{n \in \mathbb{N}} E_n$, $\phi(\bigcup_{n \in \mathbb{N}} F_n) = \phi(\bigcup_{n \in \mathbb{N}} E_n) = 0$. \square

Theorem 3. *Let ϕ be a submeasure such that $\phi(\{n\}) = 0$ for every $n \in \mathbb{N}$. Then the following conditions are equivalent.*

- (1) ϕ satisfies (A).
- (2) ϕ satisfies (A').
- (3) ϕ satisfies (B).

Proof. (1) \Leftrightarrow (2). By Proposition 1.

(2) \Rightarrow (3). Let $(A_n)_n$ be a decreasing sequence with $\lim_{n \rightarrow \infty} \phi(A_n) \neq 0$. Then there are finite sets $F_n \subset A_n$ with $\phi(\bigcup_{n \in \mathbb{N}} F_n) > 0$ (by property (A')). Let $Z = \bigcup_{n \in \mathbb{N}} F_n$. Then $Z \setminus A_n \subset F_0 \cup \dots \cup F_{n-1}$ is finite. And $\phi(Z) > 0$. This shows that ϕ satisfies (B).

(3) \Rightarrow (2). Suppose, that ϕ does not satisfy (A'). Then there is a decreasing ϕ -sequence $(A_n)_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} \phi(A_n) \neq 0$. Let $A = \bigcap_{n \in \mathbb{N}} A_n$. We have two cases.

- (1) $\phi(A) > 0$, or
- (2) $\phi(A) = 0$.

In the first case, let $A = \{a_n : n \in \mathbb{N}\}$. Let $F_n = \{a_n\}$ for every $n \in \mathbb{N}$. Then $F_n \subset A_n$ and F_n are finite. On the other hand, $\phi(\bigcup_{n \in \mathbb{N}} F_n) = \phi(A) > 0$, a contradiction.

Now, consider the second case. Then, by property (B), there is $Z \subset \mathbb{N}$ such that $\phi(Z) > 0$ and $Z \setminus A_n$ is finite (so $\phi(Z \setminus A_n) = 0$) for every $n \in \mathbb{N}$. Let $X = Z \setminus A$. Let $G_n = X \cap (A_n \setminus A_{n+1})$. Then $G_n \subset A_n$ and G_n are finite. On the other hand, $Z \subset (Z \setminus A_0) \cup \bigcup_{n \in \mathbb{N}} G_n \cup A$, so $\phi(\bigcup_{n \in \mathbb{N}} G_n) > 0$, a contradiction. \square

Remark. The assumption that ϕ vanishes on singletons is only used in the proof of “(3) \Rightarrow (2)”.

Below we will consider properties of submeasures for which the assumption of Theorem 3 does not hold.

Lemma 4. For a submeasure ϕ , let $S(\phi) = \{n \in \mathbb{N} : \phi(\{n\}) = 0\}$.

- (1) ϕ satisfies (A) $\Leftrightarrow \phi \upharpoonright \mathcal{P}(S(\phi))$ satisfies (A).
- (2) If $\phi(S(\phi)) = 0$ then ϕ satisfies (A).
- (3) If $\mathbb{N} \setminus S(\phi) \neq \emptyset$ then ϕ satisfies (B).

Proof. (1). (\Rightarrow). Obvious.

(\Leftarrow). Let $(A_n)_n$ be a ϕ -sequence. Clearly, also $(A_n \cap S(\phi))_n$ and $(A_n \setminus S(\phi))_n$ are ϕ -sequences. Since $\phi \upharpoonright \mathcal{P}(S(\phi))$ satisfies (A), $\lim_{n \rightarrow \infty} \phi(A_n \cap S(\phi)) = 0$. Since $\phi(\{k\}) > 0$ for all $k \notin S(\phi)$, $A_n \setminus S(\phi) = \emptyset$ for every n . In consequence, $\lim_{n \rightarrow \infty} \phi(A_n) = 0$.

(2). Follows from (1).

(3). Let $(A_n)_n$ be a sequence of subsets of \mathbb{N} such that $\lim_{n \rightarrow \infty} \phi(A_n) \neq 0$. Let $x \in \mathbb{N} \setminus S(\phi)$ and $Z = \{x\}$. Then $Z \setminus A_n$ is finite for every $n \in \mathbb{N}$ and $\phi(Z) > 0$. Thus ϕ satisfies (B). \square

Example 5. There is a submeasure ϕ which satisfies (B) but does not satisfy (A).

Proof. Let $\mathbb{N} \setminus \{0\} = A_0 \cup A_1 \cup \dots$ be a partition of $\mathbb{N} \setminus \{0\}$ into infinite pairwise disjoint sets. We define a submeasure ϕ by $\phi(A) = 0$ if $0 \notin A$ and $\{n \in \mathbb{N} : A \cap A_n \text{ is infinite}\}$ is finite, and $\phi(A) = 1$ otherwise.

Since $\phi(\{0\}) = 1$, so ϕ satisfies (B) (by Proposition 4).

Now, we show that ϕ does not satisfy (A). Suppose, to the contrary, that ϕ satisfies (A). Let $B_n = \bigcup_{i \geq n} A_i$. Then $\phi(B_n) = 1$ for every $n \in \mathbb{N}$. So there are finite $F_n \subset B_n$ with $\phi(\bigcup_{n \in \mathbb{N}} F_n) > 0$. Let $Z = \bigcup_{n \in \mathbb{N}} F_n$. Then $Z \cap A_n \subset F_0 \cup \dots \cup F_{n-1}$ is finite for every $n \in \mathbb{N}$. Thus $\phi(Z) = 0$, a contradiction. \square

3. A submeasure with property (A)

If \mathcal{I} is an ideal on \mathbb{N} , then $\mathcal{I}^+ = \mathcal{P}(\mathbb{N}) \setminus \mathcal{I}$ is called the *coideal associated with \mathcal{I}* .

A coideal \mathcal{I}^+ is a *P-coideal* if for every decreasing sequence $(A_n)_n$, $A_n \in \mathcal{I}^+$, there is a set $A \in \mathcal{I}^+$ such that $A \setminus A_n$ is finite for every $n \in \mathbb{N}$.

Lemma 6. Let ϕ be a submeasure which takes only two values 0 and 1, and $\phi(\{n\}) = 0$ for every $n \in \mathbb{N}$. Then the following conditions are equivalent.

- (1) ϕ satisfies (A).
- (2) ϕ satisfies (B).
- (3) $\mathcal{Z}(\phi)^+$ is a P-coideal.

Proof. The equivalence of (1) and (2) follows from Theorem 3.

(1) \Rightarrow (3). Let $(A_n)_n$ be a decreasing sequence of sets from $\mathcal{Z}(\phi)^+$. Denote $A = \bigcap_{n \in \mathbb{N}} A_n$. If $A \notin \mathcal{Z}(\phi)$, then we are done. So suppose that $A \in \mathcal{Z}(\phi)$. Let $B_n = A_n \setminus A \notin \mathcal{Z}(\phi)$. Since $\lim_{n \rightarrow \infty} \phi(B_n) = 1$ so $(B_n)_n$ is not a ϕ -sequence. Thus, there are finite sets $F_n \subset B_n$ with $\phi(\bigcup_{n \in \mathbb{N}} F_n) > 0$. Then $B = \bigcup_{n \in \mathbb{N}} F_n \notin \mathcal{Z}(\phi)$ and $B \setminus A_n \subset F_0 \cup \dots \cup F_{n-1}$ is finite for every $n \in \mathbb{N}$. Thus, $\mathcal{Z}(\phi)^+$ is a P-coideal.

(3) \Rightarrow (2). Let $(A_n)_n$ be a decreasing sequence such that $\lim_{n \rightarrow \infty} \phi(A_n) \neq 0$. Then $A_n \notin \mathcal{Z}(\phi)$ for every $n \in \mathbb{N}$. Since $\mathcal{Z}(\phi)^+$ is a P-coideal, so there is $Z \notin \mathcal{Z}(\phi)$ with $Z \setminus A_n$ finite for every $n \in \mathbb{N}$. This shows that ϕ satisfies (B). \square

Remark. The assumption that ϕ takes only two values is only used in the proof of “(1) \Rightarrow (3)”. And the assumption that ϕ vanishes on singletons is only used in the proof of “(2) \Rightarrow (1)”.

By identifying subsets of \mathbb{N} with their characteristic functions, we equip $\mathcal{P}(\mathbb{N})$ with the Cantor-space topology and therefore we can assign the topological complexity to the ideals of sets of integers. In particular, an ideal \mathcal{I} is an $F_{\sigma\delta}$ (analytic)

if it is an $F_{\sigma\delta}$ subset of the Cantor space (if it is a continuous image of a G_δ subset of the Cantor space, respectively). Moreover, an lsc submeasure is also lsc (in the topological sense) when viewed as a function on the Cantor cube.

Fact 7 (Folklore). Let ϕ be the lim sup of lsc submeasures. The ideal $\mathcal{Z}(\phi)$ is an $F_{\sigma\delta}$ subset of $\mathcal{P}(\mathbb{N})$.

Proof. We provide an argument for the completeness.

$$\mathcal{Z}(\phi) = \left\{ A \subset \mathbb{N}: \lim_{n \rightarrow \infty} \phi_n(A) = 0 \right\} = \bigcap_{k \geq 1} \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} \left\{ A \subset \mathbb{N}: \phi_n(A) \leq \frac{1}{k} \right\}$$

is $F_{\sigma\delta}$ because the families $\{A \subset \mathbb{N}: \phi_n(A) \leq 1/k\}$ are closed in $\mathcal{P}(\mathbb{N})$. \square

For an ideal \mathcal{I} , we denote by $\phi_{\mathcal{I}}$ the submeasure defined by $\phi_{\mathcal{I}}(A) = 0$ if $A \in \mathcal{I}$ and $\phi(A) = 1$ otherwise.

If we assume the Continuum Hypothesis, then it is not difficult to show an example of a submeasure with (A) which is not equivalent to the lim sup of a sequence of lsc submeasures. Namely, under the Continuum Hypothesis there exists a maximal ideal \mathcal{I} containing all finite sets such that \mathcal{I}^+ is a P-coideal (see e.g. [3] where the dual notion of p-point ultrafilters is considered). Then, by Lemma 6, $\phi_{\mathcal{I}}$ satisfies (A). Since any maximal ideal containing all finite sets does not have the Baire property (see, e.g. [1, Ch. 4, Sec. 4.1, Thm. 4.1.1]), \mathcal{I} is not $F_{\sigma\delta}$, so $\phi_{\mathcal{I}}$ is not the lim sup of lsc submeasures (by Fact 7).

In Example 8, we construct a submeasure with the above properties without any additional set theoretic assumptions.

A coideal \mathcal{I}^+ is a Q-coideal if for every $A \in \mathcal{I}^+$ and every partition $A = \bigcup_{n \in \mathbb{N}} F_n$ of A into finite sets there is $S \subset A$ such that $S \in \mathcal{I}^+$ and S intersects each F_n in one point. A coideal \mathcal{I}^+ is *selective* if it is a P-coideal and Q-coideal.

We say that a family \mathcal{A} of subsets of \mathbb{N} is *almost disjoint* if $A \cap B$ is finite for every distinct $A, B \in \mathcal{A}$.

Example 8. There is a submeasure with property (A) which is not equivalent to the lim sup of lsc submeasures.

Proof. Let \mathcal{A} be an infinite maximal almost disjoint family of infinite subsets of \mathbb{N} such that $\bigcup \mathcal{A} = \mathbb{N}$. Let $\mathcal{I}_{\mathcal{A}}$ be the ideal generated by \mathcal{A} , i.e. the family of all subsets of \mathbb{N} which can be covered by finitely many sets from \mathcal{A} (this ideal was first considered by Mathias in [4]). Let $\phi = \phi_{\mathcal{I}_{\mathcal{A}}}$.

It is known that $\mathcal{I}_{\mathcal{A}}^+$ is a P-coideal (see e.g. [5, Sec. 9, Ex. 2, Le. 1]), hence (by Lemma 6) ϕ satisfies (A).

Now, we show that ϕ is not equivalent to the lim sup of lsc submeasures. By Fact 7, it is enough to show that $\mathcal{Z}(\phi) = \mathcal{I}_{\mathcal{A}}$ is not an $F_{\sigma\delta}$ subset of $\mathcal{P}(\mathbb{N})$. But it is known that $\mathcal{I}_{\mathcal{A}}$ is not even an analytic subset of $\mathcal{P}(\mathbb{N})$.

Indeed, since $\mathcal{I}_{\mathcal{A}}^+$ is a selective coideal (see e.g. [5, Sec. 9, Ex. 2, Le. 1]), so by [5, Sec. 12, Ex. 4] if $\mathcal{I}_{\mathcal{A}}$ was an analytic ideal on \mathbb{N} then for every $B \notin \mathcal{I}_{\mathcal{A}}$ there would be an infinite $C \subset B$ such that

$$C \cap A \text{ is finite for all } A \in \mathcal{I}_{\mathcal{A}}. \quad (\star)$$

But such C is almost disjoint from any $A \in \mathcal{A}$, so by the maximality of the family \mathcal{A} it is an element of $\mathcal{A} \subset \mathcal{I}_{\mathcal{A}}$. Then $C \cap C$ is infinite, a contradiction with (\star) . \square

Remark. It can be shown that the ideal $\mathcal{I}_{\mathcal{A}}$ (from the above proof) is of the first category (hence has the Baire property).

4. A submeasure with property (C)

An ideal \mathcal{I} is *dense* if for every infinite set $A \subset \mathbb{N}$ there is an infinite set $B \subset A$ with $B \in \mathcal{I}$.

Proposition 9. Let \mathcal{I} be a dense ideal such that \mathcal{I}^+ is a Q-coideal. There is no σ -submeasure ϕ with $\mathcal{I} = \mathcal{Z}(\phi^*)$.

Proof. Suppose that there is a σ -submeasure ϕ with $\mathcal{I} = \mathcal{Z}(\phi^*)$.

Let $A_\omega = \{n \in \mathbb{N}: \phi(\{n\}) > 1\}$ and $A_k = \{n \in \mathbb{N}: \frac{1}{2^{k+1}} < \phi(\{n\}) \leq \frac{1}{2^k}\}$ for every $k \in \mathbb{N}$.

We have two cases.

- (1) There is $k \in \mathbb{N} \cup \{\omega\}$ such that A_k is infinite.
- (2) The sets A_k are finite for every $k \in \mathbb{N} \cup \{\omega\}$.

In the first case, it is not difficult to check that there is no infinite $B \subset A_k$ with $B \in \mathcal{I}$. But, \mathcal{I} is a dense ideal, a contradiction.

Now, consider the second case. Since $\phi(\mathbb{N}) > 0$ and ϕ is a σ -submeasure, so $\phi(A_\omega \cup \bigcup_{k \in \mathbb{N}} A_k) > 0$. Let $K = \{k \in \mathbb{N} \cup \{\omega\}: A_k \neq \emptyset\}$. Since \mathcal{I}^+ is a Q-coideal, there is $S \in \mathcal{I}^+$ such that $S \cap A_k = \{a_k\}$ for every $k \in K$. On the other hand,

$$\begin{aligned}
0 < \phi^\bullet(S) &= \lim_{n \rightarrow \infty} \phi(S \setminus \{0, 1, \dots, n\}) \leq \lim_{n \rightarrow \infty} \sum_{k \in K, a_k > n} \phi(\{a_k\}) \\
&= \lim_{n \rightarrow \infty} \sum_{k \in K \setminus \{\omega\}, a_k > n} \phi(\{a_k\}) \leq \lim_{n \rightarrow \infty} \sum_{k \in K \setminus \{\omega\}, a_k > n} \frac{1}{2^k} = 0,
\end{aligned}$$

a contradiction. \square

An ideal \mathcal{I} is called a *P-ideal* if for every family $\{A_n: n \in \mathbb{N}\} \subset \mathcal{I}$ there is an $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for every $n \in \mathbb{N}$. It is not difficult to check that we can assume that $A_n \subset A_{n+1}$ for every $n \in \mathbb{N}$ in the definition of a P-ideal.

Proposition 10. *If \mathcal{I} is a P-ideal containing all finite sets, then the submeasure $\phi_{\mathcal{I}}$ satisfies (C).*

Proof. Let $\phi = \phi_{\mathcal{I}}$. Let $(A_n)_n$ be such that $\lim_{n \rightarrow \infty} \phi(A_n) = 0$. Then there is $n_0 \in \mathbb{N}$ such that $A_n \in \mathcal{I}$ for every $n > n_0$. Since \mathcal{I} is a P-ideal, so there is $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for every $n > n_0$.

Let $E_n = A_n \setminus A$ for $n > n_0$. We claim that $(A_n \setminus E_n)_{n > n_0}$ is a ϕ -sequence. Indeed, let $F_n \subset A_n \setminus E_n$ be finite sets. Then

$$\bigcup_{n > n_0} F_n \subset \bigcup_{n > n_0} (A_n \setminus E_n) \subset A \in \mathcal{I},$$

so $\phi(\bigcup_{n > n_0} F_n) = 0$. \square

Theorem 11. *Assume the Continuum Hypothesis. There is a submeasure which satisfies (C) but is not equivalent to the core of a σ -submeasure.*

Proof. Let \mathcal{I} be a maximal ideal containing all finite sets such that \mathcal{I}^+ is a selective coideal (there is one under CH, see e.g. [3] where the dual notion of Ramsey ultrafilter is considered). Let $\phi = \phi_{\mathcal{I}}$.

Since \mathcal{I} is dense and \mathcal{I}^+ is a Q-coideal, so there is no σ -submeasure ψ with $\mathcal{I} = \mathcal{Z}(\psi^\bullet)$ (by Proposition 9). Thus, ϕ is not equivalent to the core of a σ -submeasure. On the other hand, $\mathcal{Z}(\phi) = \mathcal{I}$ is a P-ideal, so ϕ satisfies (C) by Proposition 10. \square

Remark. The ideal $\mathcal{I}_{\mathcal{A}}$ (from Example 8) is dense and $\mathcal{I}_{\mathcal{A}}^+$ is a selective coideal. Thus, the submeasure $\phi_{\mathcal{I}_{\mathcal{A}}}$ is not equivalent to the core of a σ -submeasure (by Proposition 9). It is not difficult to show that $\mathcal{I}_{\mathcal{A}}$ is not a P-ideal. Thus, by Proposition 12 (below), the submeasure ϕ does not satisfy (C).

Proposition 12. *Let \mathcal{I} be an ideal containing all finite sets such that \mathcal{I}^+ is a P-coideal. The submeasure $\phi_{\mathcal{I}}$ satisfies (C) \Leftrightarrow the ideal \mathcal{I} is a P-ideal.*

Proof. The part “ \Leftarrow ” follows from Proposition 10, so it is enough to show the part “ \Rightarrow ”.

Let $\phi = \phi_{\mathcal{I}}$, and let $(A_n)_{n \in \mathbb{N}}$ be an increasing sequence of sets from \mathcal{I} . Since ϕ satisfies (C) and $\lim_{n \rightarrow \infty} \phi(A_n) = 0$, so there is a subsequence (n_k) and finite $E_{n_k} \subset A_{n_k}$ such that $(A_{n_k} \setminus E_{n_k})_k$ is a ϕ -sequence. Let $F_{n_k} = E_{n_0} \cup \dots \cup E_{n_k}$. Then F_{n_k} are finite for every $k \in \mathbb{N}$ and $(A_{n_k} \setminus F_{n_k})_k$ is also a ϕ -sequence.

Let $A = \bigcup_{k \in \mathbb{N}} (A_{n_k} \setminus F_{n_k})$. If $A \in \mathcal{I}$ then $A_n \setminus A$ is finite for every $n \in \mathbb{N}$, so we are done. Thus, suppose that $A \notin \mathcal{I}$.

Let $B_{n_k} = A \setminus \bigcup_{i < k} (A_{n_i} \setminus F_{n_i})$. Then $B_{n_k} \notin \mathcal{I}$ and $B_{n_k} \supset B_{n_{k+1}}$. Since \mathcal{I}^+ is a P-coideal, so there is $B \in \mathcal{I}^+$ such that $B \setminus B_{n_k}$ is finite for every $k \in \mathbb{N}$.

Let $C = B \cap A$. Since $B = (B \cap A) \cup (B \setminus A)$ and $B \setminus A = B \setminus B_{n_0}$ is finite (hence in \mathcal{I}), so $C \in \mathcal{I}^+$.

Let $G_{n_k} = C \cap (A_{n_k} \setminus F_{n_k})$. Then $G_{n_k} \subset B \setminus B_{n_k}$ are finite and $G_{n_k} \subset A_{n_k} \setminus F_{n_k}$. Moreover, $\bigcup_{k \in \mathbb{N}} G_{n_k} = C \notin \mathcal{I}$, so $\phi(\bigcup_{k \in \mathbb{N}} G_{n_k}) > 0$. But $(A_{n_k} \setminus F_{n_k})_k$ is a ϕ -sequence, a contradiction. \square

The authors do not know if there exists any ZFC example of a submeasure with property (C) which is not equivalent to the core of a σ -submeasure. However, for the submeasure of the form $\phi_{\mathcal{I}}$ it is not very hard to check that one cannot find such an example for \mathcal{I} being a maximal ideal ($\phi_{\mathcal{I}}$ satisfies (C) iff \mathcal{I} is a P-ideal), \mathcal{I} being an F_σ ideal ($\phi_{\mathcal{I}}$ satisfies (C) iff \mathcal{I} is a P-ideal iff $\phi_{\mathcal{I}}$ is equivalent to the core of a σ -measure), or \mathcal{I} being an analytic P-ideal ($\phi_{\mathcal{I}}$ is equivalent to the core of a σ -submeasure for each \mathcal{I}).

Acknowledgments

We would like to thank the referee for shorter proofs of Proposition 1 and Lemma 4.

References

- [1] Tomek Bartoszyński, Haim Judah, *Set Theory: On the Structure of the Real Line*, A K Peters Ltd., Wellesley, MA, 1995.
- [2] Lech Drewnowski, Tomasz Łuczak, On nonatomic submeasures on \mathbb{N} . II, *J. Math. Anal. Appl.* 347 (2) (2008) 442–449.
- [3] Thomas Jech, *Set Theory*, the third millennium edition, revised and expanded, Springer Monogr. Math., Springer-Verlag, Berlin, 2003.
- [4] A.R.D. Mathias, Happy families, *Ann. Math. Logic* 12 (1) (1977) 59–111.
- [5] Stevo Todorćević, *Topics in Topology*, Lecture Notes in Math., vol. 1652, Springer-Verlag, Berlin, 1997.